An elementary definition of opetopic sets

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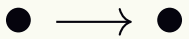
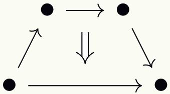
Opetopes

Geometric shapes of many-in-single-out operators in higher dimension. Used for defining weak ω-categories.

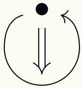
0-opetope



1-opetope



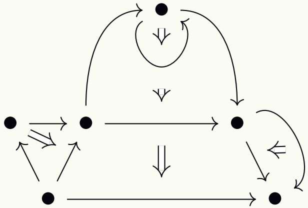
2-opetope with three sources



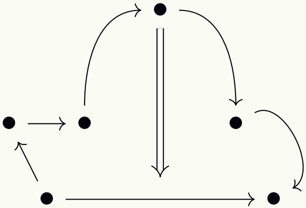
2-opetope with no source

Opetopes

A 3-opetope is determined by its pasting diagram of sources.



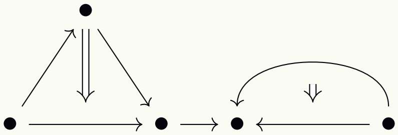
Pasting diagram of sources



Target

Opetopic sets

The opetopes form a category $\mathbb{O}$. An opetopic set is a set-valued presheaf on $\mathbb{O}$, i.e. a formal colimit of opetopes.



Formal definitions

- Baez and Dolan [1]
- Leinster [8]
- Kock, Joyal, Batanin, and Mascari [7]
- Hermida, Makkai, and Power [4] (called multitopes and multitopic sets)
- Curien, Ho Thanh, and Mimram [3]

These are not sufficiently accessible: some amount of prerequisites; too long.

Polynomial monad definition of opetopes

By Kock, Joyal, Batanin, and Mascari [7].

- A polynomial functor on I is an endofunctor on $\mathbf{Set} \downarrow I$ of the form $P(X)_i = \coprod_{b:\mathbf{B}(P)_i} \prod_{e:\mathbf{E}_P(b)} X_{\mathsf{s}_P(e)}$.
- A polynomial monad on I is a monad on $\mathbf{Set} \downarrow I$ whose underlying functor is a polynomial functor and unit and multiplication are cartesian natural transformations.
- For every polynomial monad P on I, there is a polynomial monad P^+ on $\mathbf{B}(P)$, called the **Baez-Dolan construction**, such that $\mathbf{Alg}(P^+) \simeq \mathbf{PM}_I \downarrow P$.
- The set of KJBM n-opetopes $\mathbb{O}_n^{\mathrm{KJBM}}$ and the polynomial monad Z_n on $\mathbb{O}_n^{\mathrm{KJBM}}$ are defined by $\mathbb{O}_0^{\mathrm{KJBM}} \equiv \mathbf{1}$, $\mathsf{Z}_0(X) \equiv X$, $\mathbb{O}_{n+1}^{\mathrm{KJBM}} \equiv \mathbf{B}(\mathsf{Z}_n)$, and $\mathsf{Z}_{n+1} \equiv \mathsf{Z}_n^+$.

Polynomial monad definition of opetopes

- The polynomial monad definition is a one-slide definition of opetopes, but a lot of stuff is behind the Baez-Dolan construction.
- We just get a set of opetopes. To get a category of opetopes, we must specify generators and relations. A presentation of the category of opetopes is given by Ho Thanh [5].

Contribution

I propose elementary definitions of opetopes and opetopic sets.

- Directly define the category of opetopic sets by simple structure and axiom.
- Opetopes are opetopic sets satisfying one more axiom.
- The only prerequisite is basic category theory.
- Less than two pages in A4 size.
- Equivalent to an existing one.

I can explain our definition in full detail in 30 minutes.

Foundations

We work in Univalent Foundations [9]. Constructively fine: no excluded middle; no choice axiom; no propositional resizing. Non-univalent audience may interpret types as groupoids [6] for this talk.

Gaunt categories

Definition

A category is **gaunt** if its type of objects is a set.

In non-univalent foundations, a category is gaunt if the identities are the only isomorphisms in it [2].

ω-direct categories

Definition

An ω-**direct category** is a gaunt category A equipped with a conservative functor $\mathbf{deg} : A \to \omega$ called the **degree functor**. A k-**step arrow**, written $f : x \to^k y$, is an arrow such that $\mathbf{deg}(x) + k = \mathbf{deg}(y)$. Let $\mathbf{Arr}^k(A)$ denote the set of k-step arrows. Let $A \downarrow^k x \subset A \downarrow x$ denote the subset spanned by k-step arrows into x.

Preopetopic sets

Definition

A **preopetopic set** is an ω-direct category A equipped with a subset $S(A) \subset \mathbf{Arr}^1(A)$ with complement $\mathsf{T}(A)$. A **source arrow**, written $f : x \to^{s} y$, is an arrow in $S(A)$. A **target arrow**, written $f : x \to^{t} y$, is an arrow in $\mathsf{T}(A)$.

We think of objects in a preopetopic set A as **cells**, and the arrows in A determine the configuration of the cells.

Opetopic set axioms

An opetopic set is a preopetopic set A satisfying eight axioms.

Axiom (O1)

$A \downarrow^1 x$ is finite for every $x : A$.

Each cell has finitely many sources and targets.

Definition

A set A is **finite** if there exist $n : \mathbb{N}$ and
$e : \{x : \mathbb{N} \mid x < n\} \simeq A$.

Opetopic set axioms

Axiom (O2)

For every object $x : A$ of degree ≥ 1, there exists a unique target arrow into x.

This expresses the single-out nature of opetopes.

Axiom (O3)

For every object $x : A$ of degree 1, there exists a unique source arrow into x.

This expresses that the 1-opetope $(\bullet \to \bullet)$ is single-in.

Homogeneous/heterogeneous factorizations

Definition

Let A be a preopetopic set, $f : y \to^1 x$, and $g : z \to^1 y$. We say (f, g) is **homogeneous** if either

- both f and g are source arrows; or
- both f and g are target arrows.

We say (f, g) is **heterogeneous** if either

- f is a source arrow and g is a target arrow; or
- f is a target arrow and g is a source arrow.

By a **homogeneous/heterogeneous factorization** of a 2-step arrow h we mean a factorization $h = f \circ g$ such that (f, g) is homogeneous/heterogeneous.

Opetopic set axioms

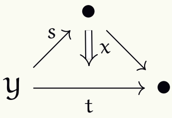
Axiom (O4)

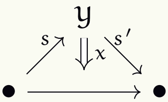
Every 2-step arrow in A has a unique homogeneous factorization.

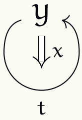
Axiom (O5)

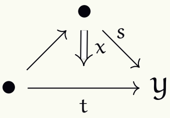
Every 2-step arrow in A has a unique heterogeneous factorization.

For example, a 0-cell y is embedded into a 2-cell x in exactly two ways, one is homogeneous and the other is heterogeneous.









Opetopic set axioms

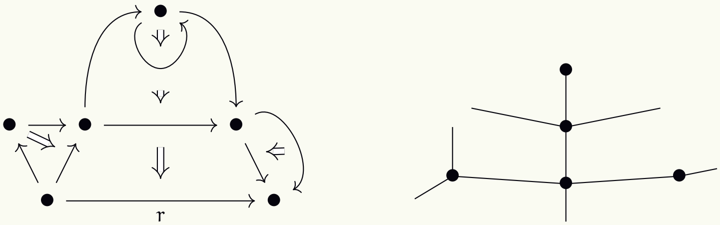
Axiom (O6)

For every object $x : A$ of degree ≥ 2, there exists a 2-step arrow $r : A \downarrow^2 x$ such that, for every 2-step arrow $f : A \downarrow^2 x$, there exists a zigzag

$$f = f_0 \xrightarrow{s_0}{}^{s} g_0 \; {}^{t}\xleftarrow{t_0} f_1 \xrightarrow{s_1}{}^{s} \cdots \xrightarrow{s_{m-1}}{}^{s} g_{m-1} \; {}^{t}\xleftarrow{t_{m-1}} f_m = r,$$

where g_i's are source arrows into x, s_i's are source arrows in $A \downarrow x$, and t_i's are target arrows in $A \downarrow x$.

Tree structures on pasting diagrams



The pasting diagram on the left has the tree structure on the right. Dots and lines in the tree correspond to 2-dimensional cells and 1-dimensional cells, respectively, in the pasting diagram.

Opetopic set axioms

A couple of global axioms.

Axiom (O7)

For every target arrow $f : y \to^{t} x$ in A and object $z : A$ of degree $\leq \mathbf{deg}(y) - 2$, the postcomposition map $f_{!} : \mathbf{Arr}_{A}(z, y) \to \mathbf{Arr}_{A}(z, x)$ is injective.

Axiom (O8)

For every $k \geq 3$, every k-step arrow $y \to^{k} x$ in A factors as $f \circ g$ such that f is a $(k-1)$-step arrow and g is a 1-step arrow.

Opetopes

Definition

An **opetope** is an opetopic set in which a terminal object exists.

Let $\mathbf{O}\mathcal{S}et$ denote the category of small opetopic sets whose morphisms are those functors preserving degrees, source arrows, and target arrows. Let $\mathbb{O} \subset \mathbf{O}\mathcal{S}et$ denote the full subcategory spanned by opetopes.

Results

- $\mathbf{O}\boldsymbol{Set} \simeq \mathbf{Psh}(\mathbb{O})$.
- Definition of pasting diagrams.
- Substitution and grafting of pasting diagrams.
- Equivalence with the polynomial monad definition by Kock, Joyal, Batanin, and Mascari [7].
- Presentation of the category of opetopes equivalent to Ho Thanh's [5].

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[1] John C. Baez and James Dolan. “Higher-dimensional algebra. III. n-categories and the algebra of opetopes”. In: *Adv. Math.* 135.2 (1998), pp. 145–206. DOI: 10.1006/aima.1997.1695.

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[3] Pierre-Louis Curien, Cédric Ho Thanh, and Samuel Mimram. “Type theoretical approaches to opetopes”. In: *Higher Structures* 6.1 (2022), pp. 80–181.

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Local rigidity

Proposition

Let $F_1, F_2 : A \to A'$ *be morphisms of opetopic sets,* $x : A$*, and* $x' : A'$ *such that* $F_1(x) = F_2(x) = x'$*. Then* $F_1 \downarrow x, F_2 \downarrow x : A \downarrow x \to A' \downarrow x'$ *are identical.*

Proposition

Let $F : A \to A'$ *be a morphism of opetopic sets and* $x : A$*. Then* $F \downarrow x : A \downarrow x \to A' \downarrow F(x)$ *is an equivalence.*

Local rigidity

Corollary

$\mathbb{O}$ is a gaunt category.

Corollary

Every morphism of opetopic sets is a discrete fibration.

Corollary

$\mathbf{OSet} \downarrow A \simeq \mathbf{Psh}(A)$ *for every* $A : \mathbf{OSet}$.

Local finiteness

Proposition

Let A be an opetopic set. Then $A \downarrow x$ is finite for every $x : A$.

Corollary

Every opetope is finite.

Corollary

$\mathbb{O}$ *is small.*

The opetopic set of opetopes

We extend $\mathbb{O}$ to a preoeptopic set.

- $\mathbf{deg}_{\mathbb{O}}(A) \equiv \mathbf{deg}_A(*_A)$, where $*_A : A$ is the terminal object.
- $F : A' \to A$ is a source/target arrow if $F(*_{A'}) \to *_A$ is a source/target arrow.

Proposition

Let A be an opetopic set. The morphism of preopetopic sets $A \to \mathbb{O} \downarrow A$ that sends $x : A$ to the forgetful functor $x_! : A \downarrow x \to A$ is an equivalence.

Corollary

$\mathbb{O}$ *is an opetopic set.*

The terminal opetopic set

Proposition

$\mathbb{O} : \mathbf{OSet}$ *is the terminal object.*

Proof.

$(x \mapsto A \downarrow x) : A \to \mathbb{O}$ is the unique morphism. □

Corollary

$\mathbf{OSet} \simeq \mathbf{Psh}(\mathbb{O})$.

The polynomial monad definition of opetopes

By Kock, Joyal, Batanin, and Mascari [7].

- A polynomial functor on I is an endofunctor on $\mathcal{S}et \downarrow I$ of the form $P(X)_i = \coprod_{b:\mathbf{B}(P)_i} \prod_{e:\mathsf{E}_P(b)} X_{\mathsf{s}_P(e)}$.
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Equivalence with the polynomial monad definition

Theorem

$\mathbb{O}_n \simeq \mathbb{O}_n^{\mathrm{KJBM}}$

Proof sketch.

Construct a polynomial monad $\mathbf{Y}_n$ on $\mathbb{O}_n$ and show that $\mathbf{Y}_0 \simeq \mathbf{Z}_0$ and $\mathbf{Y}_{n+1} \simeq \mathbf{Y}_n^+$. □

There are two compositional structures on pasting diagrams, **substitution** and **grafting**. The polynomial monad structure on $\mathbf{Y}_n$ is defined by substitution, and the equivalence $\mathbf{Y}_{n+1} \simeq \mathbf{Y}_n^+$ is proved by interaction between substitution and grafting.

Categorical equivalence

Ho Thanh [5] gives a definition of the category of opetopes, whose objects are the KJBM opetopes, by generators and relations. Our category of opetopes $\mathbb{O}$ has the following presentation, which is shown equivalent to Ho Thanh's.

Proposition

Let A be an opetopic set. Then the underlying category of A is presented by:

Generators — *all the* 1*-step arrows in* A*;*

Relations — *all the equations* $f_1 \circ g_1 = f_2 \circ g_2$ *that hold in* A *such that* (f_1, g_1) *is heterogeneous and* (f_2, g_2) *is homogeneous.*