Presentation of finite Reedy categories as localizations of finite direct categories

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Outline

Introduction

Introduction: Result

Main theorem

Theorem (Main result)

Let C be a Reedy category. We construct:

- *a direct category* $\mathrm{Down}(C)$*;*
- *an* $(\infty, 1)$*-localizing* 1*-functor* $\mathrm{Down}(C) \to C$*.*

If C is finite, so is $\mathrm{Down}(C)$*.*

This talk is based on arXiv:2502.05096.

Preceding result

A seemingly stronger result by Lurie:

Theorem (Lurie [5])

Let X be a simplicial set. Then there are:

- *a well-founded partially ordered set P;*
- *an $(\infty, 1)$-localizing simplicial map $\mathrm{N}(P) \to X$.*

If X is finite, then finite P exists.

Remark

Lurie did not state well-foundedness, which follows from construction.

- An older highly relevant result by Barwick and Kan [2, 1]:
  - a model structure on relative categories whose cofibrant objects are relative posets, giving a model of $(\infty, 1)$-categories.

Comparison

Theorem (Main result)

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If C is finite, so is $\mathrm{Down}(C)$*.*

Theorem (Lurie [5])

Let X be a simplicial set. Then there are:

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- *an* $(\infty, 1)$*-localizing simplicial map* $\mathrm{N}(P) \to X$*.*

If X is finite, then finite P exists.

The nerve $\mathrm{N}(C)$ is finite $\iff$ C is finite direct.

Introduction: Motivation

Homotopy type theory

Homotopy type theory (HoTT)

Homotopy-invariant formal theory of homotopical spaces, expressed in terms of *dependent type theory*.

type theory	homotopy theory
a type $A : \mathcal{U}$	a space A
a term $a : A$	a point $a \in A$
a dependent type $x : A \vdash B(x) : \mathcal{U}$	a fibration $B \to A$
the identity type $x, y : A \vdash x =_A y : \mathcal{U}$	the path space

Caveat: higher homotopies are luxuries: higher the homotopy, more complex the *syntax*.

Future application: finite Reedy presheaves

HARD: developping theory of space-valued presheaves

C: infinite or non-direct
$\Longrightarrow$ the nerve $\mathrm{N}(C)$: *infinite*
$\Longrightarrow$ infinitely various diagrams to commute
$\Longrightarrow$ infinitely long syntax in definition!

- Established special-case theories of space-valued presheaves (surveyed by Kraus and Sattler [4]):
  - over a finite direct category, as Reedy fibrant strict presheaves, by Shulman [6]
  - over the category freely generated by a quiver, etc.
- If C is finite Reedy:
  C-presheaf $:=$ $\mathrm{Down}(C)$-presheaf inverting some arrows.
- $\rightsquigarrow$ a definition of *finitely truncated simplicial spaces*!

Preceding work to future application

- Kraus and Sattler, in their unfinished extended abstract [4], proposed several definitions of space-valued presheaves, and claimed their equivalence[1] with proof incomplete:
  - Over a finite direct category, as weak (i.e, fully coherent $(\infty, 1)$-)functor (the number of coherence diagrams is finite).
  - General weak presheaves by resorting to infinitary logic or two-level type theory.
  - Over a finite Reedy category satisfying a certain condition, along the same line as the speaker.
- For the last item, they constructed their version of "direct replacement" $D(C)$.

[1] as long as any two of them are defined

On Kraus-Sattler's $D(C)$ [4]

- If C is Reedy and has natural number-valued degree function, then $D(C)$ is direct.
- If C is finite Reedy, then $D(C)$ is finite direct.
- The construction of $D(C)$ is much simpler than $\mathrm{Down}(C)$.
- Kraus-Sattler [4] claims:
  - the theory of C-presheaves via $D(C)$ works if $D(C) \to C$ is a Grothendieck opfibration;
  - $D(C) \to C$ is an opfibration for $C = \mathbf{\Delta}$ or $C = \mathbf{\Delta}_{\leq n}$.
- In private communication, Kraus and the speaker confirmed:
  - $D(\mathbf{\Delta}) \to \mathbf{\Delta}$ is an opfibration;
  - $D(\mathbf{\Delta}_{\leq n}) \to \mathbf{\Delta}_{\leq n}$, for $0 < n < \infty$, is NOT an opfibration;
  - if $D(C) \to C$ is an opfibration, it is $(\infty, 1)$-localizing;
  - the proof of the last claim does not apply to $D(\mathbf{\Delta}_{\leq n}) \to \mathbf{\Delta}_{\leq n}$.

Introduction: Preliminaries

Directness and Reedy structure

- A binary relation $<$ on X is *well-founded* if $Y = X$ is the unique $Y \subseteq X$ with $\forall x \in X, (\{y \mid y < x\} \subseteq Y \implies x \in Y)$.
- A category C is *direct* if the existence of non-identity $x \to y$ is a well-founded relation on $\mathrm{Ob}(C)$.
- A *Reedy structure* on a category C is a pair (C_-, C_+) where:
  - $C_-, C_+ \subseteq C$ wide subcategories;
  - For $x \to y$, $\exists!$ factorization $x \overset{-}{\twoheadrightarrow} z \overset{+}{\rightarrowtail} y$;
  - the following $<$ on $\mathrm{Ob}(C)$ is well-founded: $x < y$ if either:
    - $\exists\, x \overset{-}{\twoheadleftarrow} y$ non-identity, or
    - $\exists\, x \overset{+}{\rightarrowtail} y$ non-identity.

Simplex category

- The simplex category $\mathbf{\Delta}$: Reedy
  - Ob: $[n] := \{0 < 1 < \cdots < n\}$, $n \geq 0$
  - Mor: $[m] \to [n]$ $\leq$-preserving map
  - $\mathbf{\Delta}_{-} := \{\text{surjectives}\}$, "degeneracies"
  - $\mathbf{\Delta}_{+} := \{\text{injectives}\}$, "face maps"

Localization

- A functor $F\colon C \to D$ is *(weakly) 1-localizing* at $W \subseteq \operatorname{Mor} C$ if, for every category E:

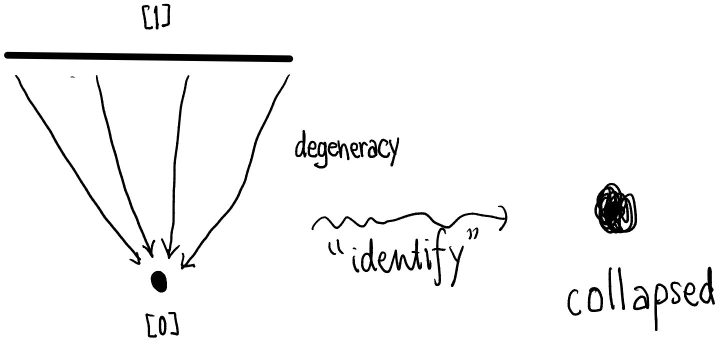
$$
\begin{array}{ccc}
\operatorname{Fun}(D, E) & \xrightarrow{\quad F^* \quad} & \operatorname{Fun}(C, E) \\
 & {\scriptstyle \text{cat. equiv.}}\ \searrow^{\exists} \qquad \nearrow & \\
 & \{C \to E \text{ inverting } W\} &
\end{array}
$$

- An $(\infty, 1)$-*localizing* simplicial map: defined similarly (for every quasi-category Q, . . .)

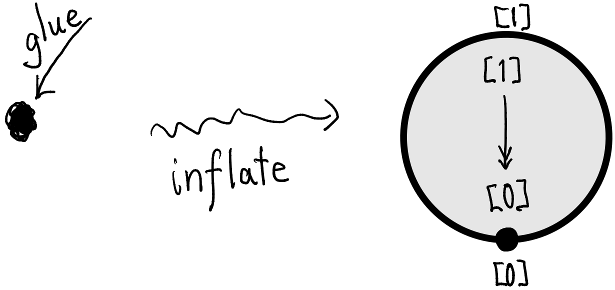
Construction of Down(C) $\to$ C

Construction: Category of chains and skew ladders

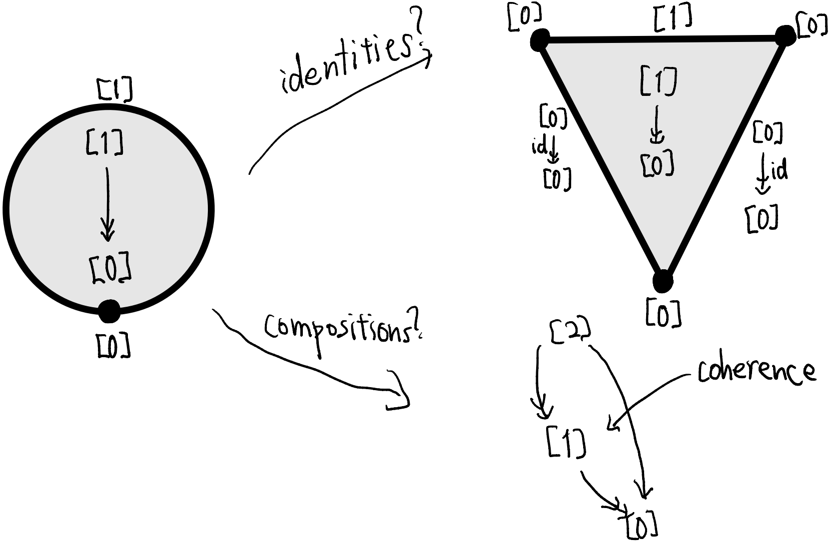
Geometric intuition: identification



Geometric intuition: glue



Geometric intuition: be categorically sane



Category of chains and skew ladders

Definition

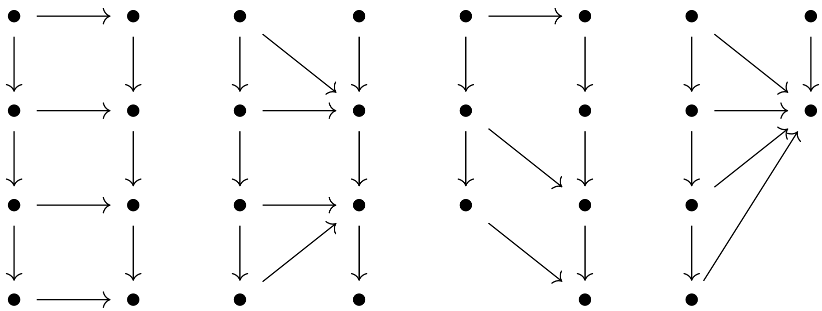
Let C be a category. The category $\int \mathbf{N}(C)$ consists of:

- objects: pairs $([n], X)$, where:
  - $[n] \in \mathrm{Ob}\,\mathbf{\Delta}$;
  - $X\colon [n] \to C$ is a functor.
- morphism $(\alpha, \theta)\colon ([m], X) \to ([n], Y)$ consists of:
  - $\alpha\colon [m] \to [n]$ in $\mathbf{\Delta}$;
  - $\theta\colon X \Rightarrow Y \circ \alpha$ a natural transformation.
- If $(\alpha, \theta)\colon ([l], X) \to ([m], Y)$ and $(\beta, \phi)\colon ([m], Y) \to ([n], Z)$, then the composite is $(\beta \circ \alpha, \xi)$ where:

$$\xi_i = \phi_{\alpha(i)} \circ \theta_i.$$

remarks on $\int \mathbf{N}(C)$

- The symbol $\int$ indicates the *Grothendieck construction*. Lumsdaine told me that the Grothendieck construction is usable here in his MathOverflow answer [3] to me.
- Example of morphisms in $\int \mathbf{N}(C)$:



Subcategories of $\int \mathbf{N}(C)$

Let C be Reedy hereafter.

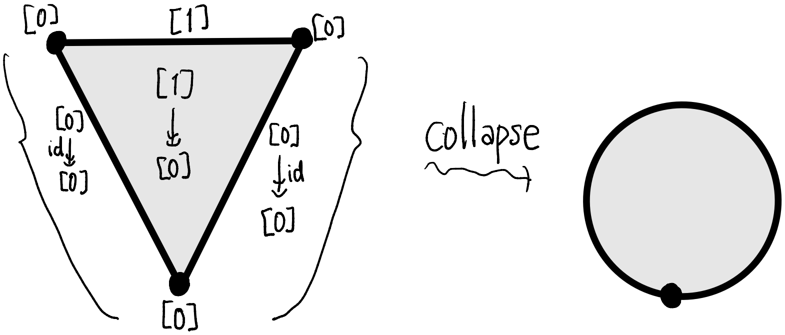
Definition

- The subcategory $\int \mathbf{N}^{-,+}(C) \subseteq \int \mathbf{N}(C)$ consists of:
  - objects $([n], X)$ where X factors through C_-;
  - morphisms (α, θ) where θ consists of morphisms in C_+.
- The full subcategory $\int \mathbf{N}_+^{--,+}(C) \subseteq \int \mathbf{N}^{-,+}(C)$ is spanned by $([n], X)$ where X reflects identities.

- The categories are due to author, but the notations (with $\int$) are completely due to Lumsdaine [3].
- $\int \mathbf{N}^{-,+}(C) \to C$ is localizing, but the category is not direct nor finite.
- $\int \mathbf{N}_+^{--,+}(C) \to C$ is NOT localizing, but the category is direct and preserves the finiteness of C.

Construction: Category of free-sliding ladders

Geometric intuition: collapse unneeded cells



Equivalence relation on morphisms I

Definition (Order of the morphisms)

Let $(\alpha, \theta), (\beta, \phi) \colon ([m], X) \to ([n], Y)$ be morphisms in $\int \mathbf{N}(C)$. We say that $(\alpha, \theta) \leq (\beta, \phi)$ if, for each $i = 0, \ldots, m$, we have $\alpha(i) \leq \beta(i)$ and the following commutative diagram:

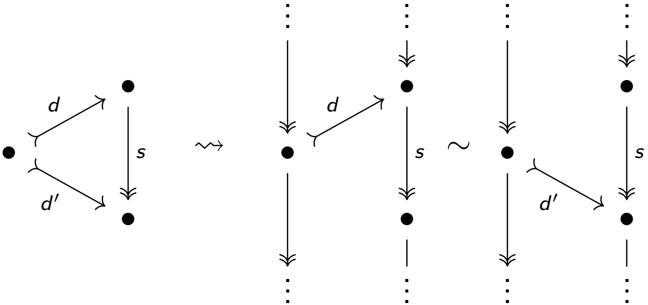
$$\begin{array}{ccc} X(i) & \xrightarrow{\theta_i} & Y(\alpha(i)) \\ & \searrow^{\phi_i} & \downarrow \\ & & Y(\beta(i)) \end{array}$$

Definition (Equivalence relation of morphisms)

Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \colon ([m], X) \to ([n], Y)$ be morphisms in $\int \mathbf{N}^{-,+}(C)$. The equivalence relation $\sim$ is given by: $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ iff there exists a common upper bound $([m], X) \to ([n], Y)$ in $\int \mathbf{N}^{-,+}(C)$.

Equivalence relation on morphisms II

The equivalence relation $\sim$ on each hom-set of $\int \mathbf{N}^{-,+}(C)$ is generated by:



Definition of Down(C)

Definition (Down(C))

The category $\mathrm{Down}_*(C)$ has the same objects as $\int \mathbf{N}^{-,+}(C)$, and its morphisms are equivalence classes of morphisms in $\int \mathbf{N}^{-,+}(C)$.
Similarly, $\mathrm{Down}(C)$ has the same objects as $\int \mathbf{N}_{+}^{--,+}(C)$, and its morphisms are equivalence classes of its morphisms.

$$\begin{array}{ccc} \int \mathbf{N}_{+}^{--,+}(C) & \xhookrightarrow{\text{full}} & \int \mathbf{N}^{-,+}(C) \\ {\scriptstyle \text{quotient}}\big\downarrow & & \big\downarrow{\scriptstyle \text{quotient}} \\ \mathrm{Down}(C) & \xhookrightarrow[\text{cat eq.}]{\sim} & \mathrm{Down}_*(C) \end{array}$$

$\mathrm{Down}(C)$ is direct and preserves the finiteness of C.

last vertex functor

Definition ($\mathfrak{last}$)

Let $\Gamma \in \{\int \mathbf{N}(C), \int \mathbf{N}^{-,+}(C), \int \mathbf{N}_{+}^{--,+}(C), \mathrm{Down}_{*}(C), \mathrm{Down}(C)\}$.

- The functor $\mathfrak{last}\colon \Gamma \to C$ is given by $\mathfrak{last}([n], X) = X(n)$.
- $f \in \mathrm{Mor}\,\Gamma$ is a $\mathfrak{last}$-*weak equivalence* if $\mathfrak{last}(f) = \mathrm{id}$.

This functor is $(\infty, 1)$-localizing but for the case $\int \mathbf{N}_{+}^{--,+}(C) \to C$.

$$\begin{array}{ccc} X(0) & & \\ \downarrow & & \\ \vdots & \xmapsto{\mathfrak{last}} & X(n) \\ \downarrow & & \\ X(n) & & \end{array}$$

Localization

Localization

Theorem

If $\Gamma \in \left\{ \int \mathbf{N}(C), \int \mathbf{N}^{-,+}(C), \mathrm{Down}_*(C), \mathrm{Down}(C) \right\}$*, then the functor* $\mathfrak{last}\colon \Gamma \to C$ *is* $(\infty, 1)$*-localizing at the* $\mathfrak{last}$*-weak equivalences.*

Given $\Gamma \to Q$ inverting $\mathfrak{last}$-weak equivalences, we need $C \to Q$.

Slogan

Do it primitively!

Example

$x \overset{s}{\twoheadrightarrow} z \overset{d}{\rightarrowtail} y$ in C should be mapped to:

$$\begin{array}{ccccccc} x & \xrightarrow{\mathrm{id}} & x & & & & \\ & & \downarrow\!\!\downarrow s & & & & \\ & & y & \xleftarrow{\mathrm{id}} & y & \xrightarrow{d} & z \end{array} \quad \text{in } \Gamma.$$

Thank you: Bibliography

References I

[1] C. Barwick and D. M. Kan. *A Thomason-like Quillen equivalence between quasi-categories and relative categories*. 2011. arXiv: 1101.0772 [math.AT]. URL: https://arxiv.org/abs/1101.0772.

[2] C. Barwick and D. M. Kan. *Relative categories: Another model for the homotopy theory of homotopy theories*. 2011. arXiv: 1011.1691 [math.AT]. URL: https://arxiv.org/abs/1011.1691.

[3] Peter LeFanu Lumsdaine. *What is the name for the construction of this poset related to coherence of degeneracies of the simplex category?* MathOverflow answer. June 4, 2022. URL: https://mathoverflow.net/q/423963 (visited on 02/06/2025).

References II

[4] Nicolai Kraus and Christian Sattler. *Space-Valued Diagrams, Type-Theoretically (Extended Abstract)*. 2017. arXiv: 1704.04543 [math.LO]. URL: https://arxiv.org/abs/1704.04543.

[5] Jacob Lurie. *Kerodon*. 2024. URL: https://kerodon.net.

[6] Michael Shulman. "Univalence for inverse diagrams and homotopy canonicity". In: *Mathematical Structures in Computer Science* 25.5 (Nov. 2014), pp. 1203–1277. ISSN: 1469-8072. DOI: 10.1017/s0960129514000565.

Localization

Localization: (weak) 1-localization

(Weak) 1-localization

The goal is to sketch:

Theorem

If Γ *is one of the four categories* $\int \mathbf{N}(C)$, $\int \mathbf{N}^{-,+}(C)$, $\mathrm{Down}_*(C)$, *and* $\mathrm{Down}(C)$, *then the functor* $\mathfrak{last}\colon \Gamma \to C$ *is weakly 1-localizing at the* $\mathfrak{last}$*-weak equivalences.*

$\Gamma = \int \mathbf{N}(C)$ is easy and is of little interest. Let Γ be any of the other three.

- $\mathfrak{last}$ sends $\mathfrak{last}$-weak equivalences to isomorphisms: trivial.
- $\mathfrak{last}^*\colon D^C \to D^\Gamma$ is fully faithful: easy. If $\epsilon\colon F \circ \mathfrak{last} \Rightarrow G \circ \mathfrak{last}$, then its unique inverse image is $\tilde{\epsilon}_x := \epsilon_{([0],x)}$.
- Factorization through $\mathfrak{last}$: hard.

Factorization through $\mathfrak{last}$ I: factor functor

Let $F\colon \Gamma \to D$ be a functor that sends $\mathfrak{last}$-weak equivalences to isomorphisms. We wish to construct a functor $G\colon C \to D$ and a natural isomorphism $\xi\colon G \circ \mathfrak{last} \Rightarrow F$.

- For each $x \in \mathrm{Ob}(C)$, set $G(x) = F([0], x)$.
- Let the following be the Reedy factorization of any morphism in C:

  $$x \xrightarrow{s} \mathrel{\mkern-14mu}\rightarrow y \rightarrowtail^{d} z$$

  We wish to set $G(d \circ s)$ as the composition of:

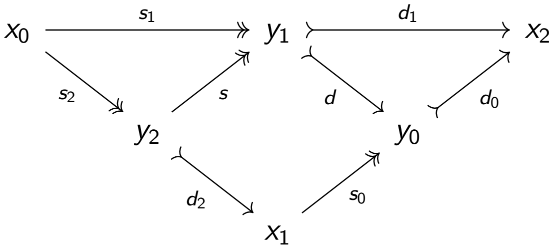
  $$F([0], x) \xrightarrow{F(\delta_1, \mathrm{id}_x)} F([1], s) \xleftarrow[\sim]{F(\delta_0, \mathrm{id}_y)} F([0], y) \xrightarrow{F(\mathrm{id}_{[0]}, d)} F([0], z).$$

  More pictorially:

  $$\begin{array}{ccccc} x & \overset{\mathrm{id}}{\rightarrowtail} & x & & \\ & & \downarrow\!\!\downarrow s & & \\ & & y & \overset{\mathrm{id}}{\leftarrowtail} & y & \overset{d}{\rightarrowtail} & z \end{array}$$

Factorization through $\mathfrak{last}$ II: functoraility I

- Let the following be the Reedy factorization of any commutative triangle $x_0 \to x_1 \to x_2$ in C:

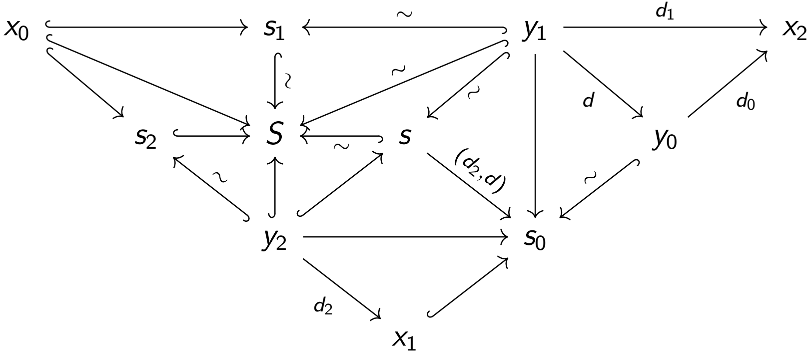


Write S for the functor $[2] \to C_-$ given by:

$$x_0 \xrightarrow{s_2} y_2 \xrightarrow{s} y_1.$$

Then the functoriality of G is shown by the commutative diagram in Γ in the next slide:

Factorization through $\mathfrak{last}$ III: functoriality II



(the length symbol from $\mathbf{\Delta}$ is omitted for simplicity.)

Factorization through $\mathfrak{last}$ IV: natural isomorphism

It remains to construct a natural isomorphism $\xi\colon G \circ \mathfrak{last} \Rightarrow F$.

- Consider $([n], X)$ in $\operatorname{Ob}\Gamma$. We have to construct:

$$\xi_{([n],X)}\colon G(X(n)) = F([0], X(n)) \to F([n], X).$$

- We set $\iota_n\colon [0] \to [n]; 0 \mapsto n$. Then we put:

$$\xi_{([n],X)} = F(\iota_n, \mathrm{id}_{X(n)}).$$

Factorization through $\mathfrak{last}$ V: naturality

- Naturality. Let $(\alpha, \theta)\colon ([m], X) \to ([n], Y)$ be a morphism in Γ. Consider:

$$
\begin{array}{ccc}
X(m) & \xrightarrow{\ s\ } \!\!\!\!\rightarrow & z \\
{\scriptstyle \theta_m} \big\downarrow & & \big\downarrow {\scriptstyle d} \\
Y(\alpha(m)) & \longrightarrow\!\!\!\!\rightarrow & Y(n)
\end{array}
$$

Then the naturality square is obtained from:

$$
\begin{array}{ccccccc}
([0], X(m)) & \hookrightarrow & ([1], s) & \hookleftarrow & ([0], z) & \xrightarrow{\ d\ } & ([0], Y(n)) \\
\big\downarrow & & & {\scriptstyle \theta_m} \searrow \ \searrow & & \searrow {\scriptstyle d} & \big\downarrow \\
([m], X) & & & \xrightarrow{\quad (\alpha,\theta) \quad} & & & ([n], Y)
\end{array}
$$

Localization: $(\infty, 1)$-localization

Wild idea

Theorem

The functor $\mathfrak{last}\colon \Gamma \to C$ *is* $(\infty, 1)$*-localizing at the* $\mathfrak{last}$*-weak equivalences for the same set of* Γ*:* $\int \mathbf{N}(C)$, $\int \mathbf{N}^{-,+}(C)$, $\mathrm{Down}_*(C)$, *and* $\mathrm{Down}(C)$.

Slogan

Primitively generalize the proof of 1-localization to higher simplices.

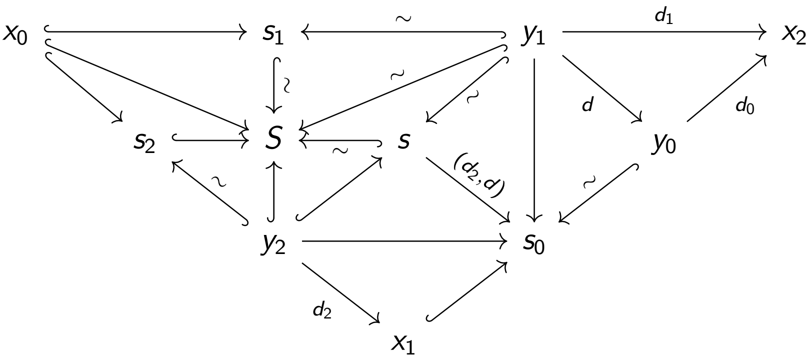
We have used many diagrams in the proof of 1-localization.

diagrams used for constructing factor functor

- 1-simplex:

$$([0], x) \xrightarrow{(\delta_1, \mathrm{id}_x)} ([1], s) \xleftarrow[\sim]{(\delta_0, \mathrm{id}_y)} ([0], y) \xrightarrow{(\mathrm{id}_{[0]}, d)} ([0], z).$$

- 2-simplex:

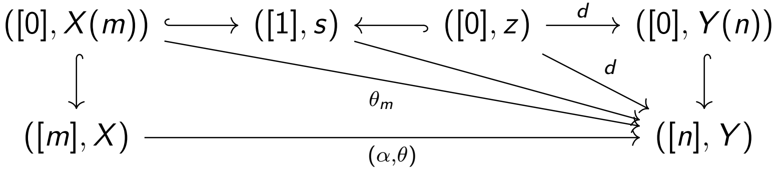


diagrams used for constructing natural isom

- 0-simplex:

$$(\iota_n, \mathrm{id}_{X(n)}) \colon ([0], X(n)) \to ([n], X).$$

- 1-simplex:



Winding up to a simplicial map

- These diagrams glues together to form a simplicial map:
  - For functor: $\mathrm{Dcp}(N(C)) \to N(\Gamma)$;
  - For natural isomorphism: $\mathrm{DcpI}(N(\Gamma)) \to N(\Gamma)$.

  For some colimit-preserving endofunctors Dcp and DcpI on $\mathbf{Set}_{\mathbf{\Delta}}$.
- If $\mathrm{Dcp}\, X$ has a localization that has X as a simplicial subset, we can leverage $\mathrm{Dcp}(N(C)) \to N(\Gamma)$ to $N(C) \to N(\Gamma)$ under a condition.
- Likewise, if $\mathrm{DcpI}\, X$ has a localization that has $X \times \Delta[1]$ as a simplicial subset, we can leverage $\mathrm{DcpI}(N(\Gamma)) \to N(\Gamma)$ to $N(\Gamma) \times \Delta[1] \to N(\Gamma)$ under a condition.